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Lifting up a tree of prime ideals to a going-up extension☆

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Abstract

We prove that if $R \subseteq D$ is an extension of commutative rings with identity and the going-up property (for example, an integral extension), then any tree $\mathcal{T}$ of prime ideals of R can be embedded in $Spec(D)$, i.e., $\mathcal{T}$ can be covered by some isomorphic tree $\mathcal{T}'$ of prime ideals of D. In particular, the prime spectrum of a Prüfer domain can always be embedded in the prime spectrum of its integral extension. The most interesting case is when the integral extension is also a Prüfer domain. In this case, we obtain two Prüfer domains such that $Spec(R) \hookrightarrow Spec(D)$. We also prove that for an integral domain R, there exists a Bézout domain D such that any tree $\mathcal{T} \subseteq Spec(R)$ can be embedded in $Spec(D)$. We give a sufficient condition for the question: given an extension $A \subseteq B$ of commutative rings and a tree $\mathcal{T} \subseteq Spec(B)$, what are necessary and sufficient conditions that $\mathcal{T}^c = \{Q \cap A | Q \in \mathcal{T}\}$ be a tree in $Spec(A)$? We also prove that if R is an integral domain with the following property: for a given tree $\mathcal{T}$ in $Spec(R)$, there exists a Prüfer overring $P(R)$ of R with the tree $\mathcal{T}'$ such that $(\mathcal{T}')^c = \mathcal{T}$ and $\mathcal{T} \cong \mathcal{T}'$, then an integral and mated extension of R has the same property.

MSC: 13A15; 13B24; 13G05

1. Introduction

Let $R \subseteq D$ be an extension of commutative rings with identity and let $\mathcal{T}$ be a subset of $Spec(R)$. We say that $\mathcal{T}$ can be lifted to $Spec(D)$ if there exists a subset $\mathcal{S}$ of

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Spec(D) lying over $\mathscr{T}$. If that happens, we say that $\mathscr{S}$ lies over or covers $\mathscr{T}$. In case when $\mathscr{S}$ can be chosen to be isomorphic to $\mathscr{T}$ as a partially ordered set, we say that $\mathscr{T}$ can be embedded in *Spec*(D). In this paper, we study when lifting or embedding is possible. It is well known that for a finite chain of prime ideals in an integral domain R, there exists a valuation overring and a chain of prime ideals lying over the given chain [10, Corollary 19.7], which has many applications in commutative ring theory. This result has been extended to an arbitrary chain of prime ideals by the authors [12, Theorem]. As a corollary, it follows that an arbitrary chain of prime ideals of R can be lifted to the integral closure in its quotient field [12, Corollary]. There have been attempts to generalize this result to a going-up extension of a commutative ring with identity. For example, Dobbs and Fontana showed that (1) any tree of prime ideals with two branches that are well ordered with respect to inclusion or in which each branch has finite length can be lifted to a tree in a going-up extension [7, Theorem 3; 8] and that (2) any chain of prime ideals that are well ordered with respect to inclusion can be lifted to a chain in a going-up extension [4, Theorem; 5].

In view of the fact that the globalization of a valuation domain is a Prüfer domain and the spectrum of a Prüfer domain forms a tree, it is natural to ask whether (♯) a tree of prime ideals of an integral domain can be lifted to a Prüfer extension [1, Problem 7]. According to Dobbs and Fontana [7, Example 8(b)], (♯) fails if the Prüfer extension is to be taken as an overring. However, (♯) holds for a tree with well-ordered branches.

The main purpose of this paper is to establish the most generalized form of the above results, namely to prove the following statements:

($*$) any tree of prime ideals can be naturally embedded in the spectrum of a going-up extension and,

($**$) for any integral domain R there exists a Bézout domain D such that every tree in *Spec*(R) can be embedded in *Spec*(D).

These provide answers to the questions posed by Anderson [1, Problem 7]. In Section 2, we introduce a CLO extension $R \subseteq D$, which is defined as an extension such that any chain in *Spec*(R) can be covered by a suitable chain in *Spec*(D). We introduce an SCLO extension that is an extension such that any chain of prime ideals ascending from P with a prime ideal Q lying over P can be covered by a chain of prime ideals ascending from Q. Then we show that an SCLO extension is both a CLO extension and a GU extension. However, a CLO extension need not be an SCLO extension. We show that every tree of prime ideals can be embedded in the spectrum of an SCLO extension. In Section 3, we obtain one of our main results ($*$) by proving that the 'SCLO extension' is identical with the 'GU extension.' We present an example that shows that outside the class of the GU extensions, ($*$) need not hold. Another example is constructed to demonstrate that even in an integral extension, a subset of prime ideals need not be embedded in the integral extension if the subset is not a tree. An interesting application of ($*$) is that for two Prüfer domains $R \subseteq D$, *Spec*(R) can be embedded in *Spec*(D) provided that D is an integral extension of R. In Section 4, using the integral extension and the Kronecker function ring we achieve another main result ($**$): for an integral domain R, the Kronecker function ring $D = (R')^b$ is a Bézout ring and every tree of prime ideals of R can be embedded in $Spec(D)$. In the final section we consider various situations to decide when a tree of prime ideals can contract to a tree. Unlike lifting, a tree of prime ideals need not contract to a tree even in an extension which satisfies both going-up and going-down property (Example 20).

2. Lifting up a tree of prime ideals to an SCLO extension

In this paper, a commutative ring means a commutative ring with identity. We begin with some definitions and basic properties.

Definition. (1) A partially ordered set $\mathscr{T}$ is a tree if for any two non-comparable elements $x, y \in \mathscr{T}$, x, y do not have an upper bound in $\mathscr{T}$.

(2) Two trees $\mathscr{T}_1$ and $\mathscr{T}_2$ are isomorphic if they are isomorphic as partially ordered sets, i.e., if there is an order-preserving bijection from $\mathscr{T}_1$ to $\mathscr{T}_2$ such that the inverse function also preserves order. In this case we write $\mathscr{T}_1 \cong \mathscr{T}_2$.

(3) A subset $\mathscr{B}$ of a tree $\mathscr{T}$ is called a branch of $\mathscr{T}$ if $\mathscr{B}$ is a linearly ordered subset of $\mathscr{T}$ and there is no linearly ordered subset of $\mathscr{T}$ properly containing $\mathscr{B}$.

Since we are mainly concerned with prime ideals, we consider the partially ordered set $Spec(R)$, where R is a commutative ring with identity. A special case of a tree in $Spec(R)$ is a chain, namely a linearly ordered subset of $Spec(R)$. Evidently, a nonempty subset $\mathscr{T}$ of $Spec(R)$ is a chain if and only if $\mathscr{T}$ has a unique branch. Lemmas 1 and 2 collect some basic properties.

Lemma 1 (Dobbs and Fontana [7, Lemma 1]). *Let R be a commutative ring and $\mathscr{T}$ be a nonempty tree in $Spec(R)$. Then*

(1) *If $\mathscr{C}$ is a chain and $\mathscr{C} \subseteq \mathscr{T}$, then there exists at least one branch $\mathscr{B}$ of $\mathscr{T}$ such that $\mathscr{C} \subseteq \mathscr{B}$.*

(2) *$\mathscr{T}$ is the union of its branches.*

Next we show that an element in a tree that is smaller than an element in a branch is in fact already contained in the branch.

Lemma 2. *Let $\mathscr{T}$ be a tree of prime ideals of a commutative ring R and $\mathscr{B}$ be a branch of $\mathscr{T}$. If $Q \in \mathscr{T}$ and $Q \subseteq P$ for some $P \in \mathscr{B}$, then $Q \in \mathscr{B}$.*

Proof. Suppose that $Q \in \mathscr{T}$ and $Q \subseteq P$ for some $P \in \mathscr{B}$. Then P is an upper bound of Q and P', where P' is an arbitrary prime ideal of R such that $P' \subseteq P$ and $P' \in \mathscr{B}$. Hence $\mathscr{T}$ being a tree implies that Q is comparable with each $P' \in \mathscr{B}$ such that $P' \subseteq P$. So Q is comparable with each element in $\mathscr{B}$. Since $Q \in \mathscr{T}$ and $\mathscr{B}$ is a branch in $\mathscr{T}$, $Q \in \mathscr{B}$. □

Definition. Let $R \subseteq D$ be an extension of commutative rings and $\mathscr{C} \subseteq Spec(R)$ be any chain of prime ideals of R. If there exists a chain $\mathscr{C}'$ of prime ideals of D which is lying-over $\mathscr{C}$, then we say that the extension $R \subseteq D$ has the *CLO-property* and $R \subseteq D$ is called a *CLO* extension. If $R/(Q \cap R) \subseteq D/Q$ has the CLO-property for any prime ideal Q in *Spec*(D), then we say that $R \subseteq D$ has the *Strong CLO-property* (for short, *SCLO-property*).

From the definition, the next lemma immediately follows.

Lemma 3. *Let $R \subseteq D$ be an extension of commutative rings. Then $R \subseteq D$ has the SCLO-property if and only if the following property holds: If $\mathscr{C}$ is a chain of prime ideals of R with the initial element P and Q is a prime ideal of D lying over P, then there exists a chain $\mathscr{C}'$ of prime ideals of D lying over $\mathscr{C}$ whose initial element is Q.*

We begin with some basic properties on CLO extensions. If a ring extension $R \subseteq D$ has the going-up property, then we say that $R \subseteq D$ is a GU extension.

Proposition 4. *Let $R \subseteq D$ be an extension of commutative rings. Then the following hold:*

(1) *If $R \subseteq D$ is an SCLO extension, then $R \subseteq D$ is a GU extension.*
(2) *If $R \subseteq D$ is an SCLO extension, then $R \subseteq D$ is a CLO extension.*

Proof. (1) Let $P_1 \subseteq P_2$ be prime ideals of R and Q_1 be a prime ideal of D lying over P_1. Since $R/P_1 \subseteq D/Q_1$ is a CLO extension and hence a lying over extension, there exists a prime ideal Q_2/Q_1 of D/Q_1 lying over P_2/P_1. Then Q_2 is the desired prime ideal of D lying over P_2.

(2) Note that a GU extension is an LO extension [13, Theorem 42]. Let $\mathscr{C}$ be a chain in *Spec*(R) and $P := \cap_{A \in \mathscr{C}} A$. Choose a prime ideal Q of D lying over P. Now the conclusion follows from Lemma 3: choose a chain $\mathscr{C}''$ lying over $\mathscr{C} \cup \{P\}$ with the initial element Q. Let

$$\mathscr{C}' := \begin{cases} \mathscr{C}'' \setminus \{B | B \in \mathscr{C}'', B^c = P\} & \text{if } P \notin \mathscr{C}, \\ \mathscr{C}'' & \text{if } P \in \mathscr{C}. \end{cases}$$

Then $\mathscr{C}'$ is the desired chain lying over $\mathscr{C}$. $\square$

Proposition 5. *Let $R \subseteq D$ be an extension of commutative rings. If $R \subseteq D$ is an integral extension, then $R \subseteq D$ has the SCLO-property.*

Proof. For a prime ideal Q of D, D/Q is integral over $R/(Q \cap R)$. So we may assume that R and D are integral domains. Let $\mathscr{C}$ be a chain in *Spec*(R). By [12, Theorem], there exists a valuation overring V of R with a chain $\mathscr{C}'$ of prime ideals lying over $\mathscr{C}$. Since the quotient field L of D is algebraic over the quotient field of R, there exists a valuation extension W of V to L such that $W \supseteq D$ by [10, Theorem 20.1]. By [10, Theorem 19.16 (b)], there exists a chain $\mathscr{C}''$ in *Spec*(W) lying over $\mathscr{C}'$. Then $\mathscr{C}'' \cap D$ is the desired chain in *Spec*(D) lying over $\mathscr{C}$ (note that $W \supseteq D$). $\square$

Later we will give an example of an SCLO extension that is not an integral extension and that has SCLO-property. Before presenting our main result, we will treat the case with only one branch.

Proposition 6. (1) *Let $R \subseteq D$ be an extension of commutative rings satisfying CLO-property. If $\mathscr{C}$ is a chain of prime ideals of R, then there exists a chain $\mathscr{C}'$ of prime ideals of D such that $(\mathscr{C}')^c = \mathscr{C}$ and $\mathscr{C}' \cong \mathscr{C}$ via the contraction.*

(2) *Let $R \subseteq D$ be an extension of commutative rings satisfying SCLO-property. If $\mathscr{C}$ is a chain of prime ideals of R with the initial element P and Q is a prime ideal of D lying over P, then there exists a chain $\mathscr{C}'$ of prime ideals of D lying over $\mathscr{C}$ whose initial element is Q such that $\mathscr{C}' \cong \mathscr{C}$ via the contraction.*

Proof. (1) Let $\mathscr{C} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a given chain of prime ideals of R. Since $R \subseteq D$ has the CLO-property, we can choose a chain $\mathscr{T}'$ of prime ideals of D lying over $\mathscr{C}$. Let $\mathscr{T}'_{P_\alpha} = \{Q \in \mathscr{T}' | Q^c = P_\alpha\}$. Note that $\mathscr{T}'_{P_\alpha} \cap \mathscr{T}'_{P_\beta} = \phi$ if and only if $P_\alpha \neq P_\beta$. By the Axiom of Choice, we can choose one Q_α in $\mathscr{T}'_{P_\alpha}$. Put $\mathscr{C}' = \{Q_\alpha\}_{\alpha\in\Lambda}$. Then $(\mathscr{C}')^c = \mathscr{C}$ and $\mathscr{C}' \cong \mathscr{C}$.

(2) The proof is similar to (1). □

Now we give a main result. In [7, Theorem 3; 8], Dobbs and Fontana proved that if an extension $R \subseteq D$ of commutative rings satisfies the going-up property, then any tree of prime ideals of R with two branches which are well ordered with respect to inclusion or any tree in which each branch has finite length is covered by some corresponding tree of prime ideals of D. We generalize this result to an arbitrary tree when $R \subseteq D$ has the SCLO-property. Moreover, the covering tree can be chosen isomorphic to the underlying tree.

Theorem 7. *Let $R \subseteq D$ be an extension of commutative rings with the SCLO-property. If $\mathscr{T}$ is a tree of prime ideals of R, then there exists a tree $\mathscr{T}'$ of prime ideals of D such that $(\mathscr{T}')^c = \mathscr{T}$ and $\mathscr{T}' \cong \mathscr{T}$ via the contraction.*

Proof. Let $\{\mathscr{B}_\alpha | \alpha \in \Lambda\}$ be the set of all branches of $\mathscr{T}$. By Lemma 1 (2), $\mathscr{T} = \bigcup_{\alpha\in\Lambda}\mathscr{B}_\alpha$. For a subset $\Gamma \subseteq \Lambda$, $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ is also a tree. Let $\mathscr{S} := \{(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha, \mathscr{T}') | \Gamma \subseteq \Lambda\}$, where $\mathscr{T}'$ is a tree in $Spec(D)$ such that $(\mathscr{T}')^c = \bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ and $(\mathscr{T}')^c \cong \mathscr{T}'$ via the contraction. By Propositions 4 and 6, $\mathscr{S}$ is a nonempty set. We define an order $\leqslant$ on $\mathscr{S}$ as following: $(\bigcup_{\alpha\in\Gamma_1}\mathscr{B}_\alpha, \mathscr{T}'_1) \leqslant (\bigcup_{\alpha\in\Gamma_2}\mathscr{B}_\alpha, \mathscr{T}'_2)$ if and only if $\Gamma_1 \subseteq \Gamma_2$ and $\mathscr{T}'_1 \subseteq \mathscr{T}'_2$.

Clearly $\mathscr{S}$ is a partially ordered set.

Let $\mathscr{C} = \{(\bigcup_{\alpha\in\Gamma_\beta}\mathscr{B}_\alpha, \mathscr{T}'_\beta)\}_{\beta\in\Phi}$ be an arbitrary chain in $\mathscr{S}$. For simplicity of index, put $\Gamma_\Phi = \bigcup_{\beta\in\Phi}\Gamma_\beta$. We claim that $(\bigcup_{\alpha\in\Gamma_\Phi}\mathscr{B}_\alpha, \bigcup_{\beta\in\Phi}\mathscr{T}'_\beta)$ is an upper bound of the above chain in $\mathscr{S}$. Since $\{\mathscr{T}'_\beta\}_{\beta\in\Phi}$ is linearly ordered and $\mathscr{T}'_\beta$ is a tree in $Spec(D)$ for each $\beta \in \Phi$, $\bigcup_{\beta\in\Phi}\mathscr{T}'_\beta$ is also a tree in $Spec(D)$. Since $(\mathscr{T}'_\beta)^c = \bigcup_{\alpha\in\Gamma_\beta}\mathscr{B}_\alpha$, we have $(\bigcup_{\beta\in\Phi}\mathscr{T}'_\beta)^c = \bigcup_{\beta\in\Phi}(\mathscr{T}'_\beta)^c = \bigcup_{\beta\in\Phi}(\bigcup_{\alpha\in\Gamma_\beta}\mathscr{B}_\alpha) = \bigcup_{\alpha\in\Gamma_\Phi}\mathscr{B}_\alpha$. Hence $(\bigcup_{\alpha\in\Gamma_\Phi}\mathscr{B}_\alpha, \bigcup_{\beta\in\Phi}\mathscr{T}'_\beta)$ is an upper bound of the given chain $\mathscr{C}$ in $\mathscr{S}$.

By Zorn's lemma, $\mathscr{S}$ has a maximal element, say $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha, \mathscr{T}')$.

We claim that $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha=\mathscr{T}$. Suppose that $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha\subsetneq\mathscr{T}$. Choose $P\in\mathscr{T}\setminus\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ and a branch $\mathscr{B}_\beta$ of $\mathscr{T}$ such that $P\in\mathscr{B}_\beta$. (Hence $\beta\notin\Gamma$.)

Suppose $\mathscr{B}_\alpha\cap\mathscr{B}_\beta=\phi$ for each $\alpha\in\Gamma$. Choose a chain $\mathscr{B}'_\beta$ of prime ideals of D lying over $\mathscr{B}_\beta$ (Propositions 4 and 6). Let $\phi: Spec(D)\to Spec(R)$ be the map induced by the contraction. Clearly ϕ is a bijection from $\mathscr{T}'\cup\mathscr{B}'_\beta$ onto $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha\cup\mathscr{B}_\beta$. We show that ϕ is a poset isomorphism. Suppose $\phi(a)\geqslant\phi(b)$ for $a,b\in\mathscr{T}'\cup\mathscr{B}'_\beta$. Choose a branch $\mathscr{B}_\gamma$ of $\mathscr{T}$ containing $\phi(a)$ and $\phi(b)$. Then $\mathscr{B}_\gamma=\mathscr{B}_\alpha$ for some $\alpha\in\Gamma$ or $\mathscr{B}_\gamma=\mathscr{B}_\beta$. In either case, we have $a\geqslant b$ since, by assumption, ϕ is an isomorphism on $\mathscr{T}'$ as well as on $\mathscr{B}'_\beta$. Thus $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha\cup\mathscr{B}_\beta,\mathscr{T}'\cup\mathscr{B}'_\beta)\in\mathscr{S}$. Then $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha\cup\mathscr{B}_\beta,\mathscr{T}'\cup\mathscr{B}'_\beta)>(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha,\mathscr{T}')$, which contradicts the maximality of $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha,\mathscr{T}')$. Hence, for some $\alpha\in\Gamma$, $\mathscr{B}_\alpha\cap\mathscr{B}_\beta$ is a nonempty (linearly ordered) subset of $\mathscr{B}_\alpha$. Then $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha)\cap\mathscr{B}_\beta$ is a nonempty linearly ordered subset of both $\mathscr{B}_\beta$ and $(\mathscr{T}')^c$. Put $(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha)\cap\mathscr{B}_\beta=\{P_\gamma|\gamma\in\Theta\}$ and $P_0=\bigcup_{\gamma\in\Theta}P_\gamma$, which is the supremum of the meeting points of $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ and $\mathscr{B}_\beta$. Note that P_0 need belong neither to $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ nor to $\mathscr{B}_\beta$. We will show that $P_0\subseteq P$ for any $P\in\mathscr{B}_\beta\setminus\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$. Since each $P_\gamma\in\mathscr{B}_\beta$, P and P_γ are comparable. If $P\subseteq P_\gamma$ for some $\gamma\in\Theta$, where $P_\gamma\in\mathscr{B}_\alpha$, then by Lemma 2, $P\in\mathscr{B}_\alpha$ for some $\alpha\in\Gamma$. This contradicts $P\notin\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$. So $P\supset P_\gamma$ for any γ. Hence $P\supseteq\bigcup_{\gamma\in\Theta}P_\gamma$. Since $\mathscr{T}'\cong(\mathscr{T}')^c$, we can choose a chain $\{Q_\gamma|\gamma\in\Theta\}$ of prime ideals of D in $\mathscr{T}'$ which is lying over $\{P_\gamma|\gamma\in\Theta\}$. Since P_0 need not belong to $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$, we define a prime ideal $Q_0\in D$ such that $Q_0\cap R=P_0$ in two ways according to $P_0\in\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ or $P_0\notin\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$. If $P_0\in\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$, then we can choose $Q_0\in\mathscr{T}'$ such that $Q_0\cap R=P_0$ since $(\mathscr{T}')^c=\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$. Let $\phi: Spec(D)\to Spec(R)$ be the map induced by the contraction. Since each $Q_\gamma\in\mathscr{T}'$, $\phi(Q_\gamma)\subseteq\phi(Q_0)$, and ϕ is an isomorphism on $\mathscr{T}'$, we have $Q_\gamma\subseteq Q_0$. So $\bigcup_{\gamma\in\Theta}Q_\gamma\subseteq Q_0$. If $P_0\notin\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$, then let $Q_0:=\bigcup_{\gamma\in\Theta}Q_\gamma$. Then we also have $Q_0\cap R=P_0$.

In either case, $Q_0\cap R=P_0$ and $\bigcup_{\gamma\in\Theta}Q_\gamma\subseteq Q_0$. (This is crucial in several critical arguments. The reason why we define Q_0 in two different ways will be illuminated naturally in the process of proving several claims.)

Let $\mathscr{B}_\beta(P_0):=\{P'\in\mathscr{B}_\beta|P_0\subseteq P'\}$. Now $\mathscr{C}:=\mathscr{B}_\beta(P_0)\cup\{P_0\}$ is a chain in $Spec(R)$ with the initial element P_0. Since $R\subseteq D$ is an SCLO extension, there exists a chain $\mathscr{C}'$ in $Spec(D)$ with the initial element Q_0 such that $\mathscr{C}'\cong\mathscr{C}$ via the contraction and $(\mathscr{C}')^c=\mathscr{C}$ by Proposition 6 (2). Define

$$\mathscr{B}'_\beta(Q_0) := \begin{cases} \mathscr{C}' & \text{if } P_0\in\mathscr{T} \text{ (in this case, } P_0\in\mathscr{B}_\beta(P_0)\text{ since } P_0\subseteq P,\ P\in\mathscr{B}_\beta),\\ \mathscr{C}'\setminus\{Q_0\} & \text{if } P_0\notin\mathscr{T} \text{ (in this case, } P_0\notin\mathscr{B}_\beta(P_0)\text{ since } P_0\notin\mathscr{B}_\beta).\end{cases}$$

Then $P_0\in\mathscr{B}_\beta(P_0)\Leftrightarrow Q_0\in\mathscr{B}'_\beta(Q_0)$. So $\mathscr{B}'_\beta(Q_0)$ is a chain of prime ideals of D such that $\mathscr{B}'_\beta(Q_0)\cong\mathscr{B}_\beta(P_0)$ and $\mathscr{B}'_\beta(Q_0)^c=\mathscr{B}_\beta(P_0)$. Let $\mathscr{T}'_1:=\mathscr{T}'\cup\mathscr{B}'_\beta(Q_0)$. Then $(\mathscr{T}'_1)^c=(\mathscr{T}')^c\cup\mathscr{B}_\beta(P_0)=(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha)\cup\mathscr{B}_\beta(P_0)=(\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha)\cup\mathscr{B}_\beta$: (If $P'\subsetneq P_0$ and $P'\in\mathscr{B}_\beta$, then $P'\in(\mathscr{T}')^c=\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$: The argument goes like this. Since $P_0=\bigcup_{\gamma\in\Theta}P_\gamma$, where $\{P_\gamma\}$ is the set of all meeting points of $\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha$ and $\mathscr{B}_\beta$, P', which is smaller than P_0, has to be smaller than some P_γ. Let $P_\gamma\in\mathscr{B}_{\alpha_0}(\alpha_0\in\Gamma)$. Then Lemma 2 implies that $P'\in\mathscr{B}_{\alpha_0}\subseteq\bigcup_{\alpha\in\Gamma}\mathscr{B}_\alpha=(\mathscr{T}')^c$.)

Finally we show that $(\mathscr{T}'_1)^c \cong \mathscr{T}'_1$.

We will show that ϕ is a bijection from $\mathscr{T}'_1$ onto $(\mathscr{T}'_1)^c = (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha) \cup \mathscr{B}_\beta$.

Case I: $P_0 \in \mathscr{T}$. Note that $(\mathscr{T}'_1)^c = (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha \setminus \{P_0\}) \overset{o}{\cup} (\mathscr{B}_\beta(P_0) \setminus \{P_0\}) \overset{o}{\cup} \{P_0\}$, where $\overset{o}{\cup}$ is a disjoint union. Since $\mathscr{T}' \cong \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$, $\mathscr{T}' \setminus \{Q_0\} \cong \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha \setminus \{P_0\}$. (If $P_0 \in \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$, then recall that we take $Q_0 \in \mathscr{T}'$ to begin with. If not, then $Q_0 \notin \mathscr{T}'$ automatically since $(\mathscr{T}')^c = \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$.) Also clearly $\mathscr{B}_\beta(P_0) \setminus \{P_0\} \cong \mathscr{B}'_\beta(Q_0) \setminus \{Q_0\}$. Hence we have $\mathscr{T}'_1 = (\mathscr{T}' \setminus \{Q_0\}) \overset{o}{\cup} (\mathscr{B}'_\beta(Q_0) \setminus \{Q_0\}) \overset{o}{\cup} \{Q_0\}$, $(\mathscr{T}' \setminus \{Q_0\})^c = \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha \setminus \{P_0\}$, $(\mathscr{B}'_\beta(Q_0) \setminus \{Q_0\})^c = \mathscr{B}_\beta(P_0) \setminus \{P_0\}$, and $(Q_0)^c = P_0$. Also note that the contraction function and its inverse are bijections on each of the disjoint three subsets of $\mathscr{T}'_1$. Moreover the images of these three sets are also disjoint with each other. Hence $\phi : \mathscr{T}'_1 \to (\mathscr{T}'_1)^c$ is a bijection.

Case II: $P_0 \notin \mathscr{T}$. In this case, $P_0 \notin \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$ (see the definition of P_0). So $\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$ and $\mathscr{B}_\beta(P_0)$ are disjoint. Since $\mathscr{T}' \cong \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$ and $\mathscr{B}'_\beta(Q_0) \cong \mathscr{B}_\beta(P_0)$, we have $\mathscr{T}'_1 = \mathscr{T}' \overset{o}{\cup} \mathscr{B}'_\beta(Q_0)$. Hence ϕ is a bijection from $\mathscr{T}'_1$ onto $(\mathscr{T}'_1)^c = (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha) \cup \mathscr{B}_\beta(P_0) = (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha) \cup \mathscr{B}_\beta$.

Thus in either case, ϕ is a bijection. So, in order to prove that $\mathscr{T}'_1 \cong (\mathscr{T}'_1)^c$, it suffices to show that the inverse function of the contraction function preserves the partial order, namely the inclusion.

Let $P_1 \subseteq P_2$, where $P_1, P_2 \in (\mathscr{T}'_1)^c$. Choose $Q_1, Q_2 \in \mathscr{T}'_1$ such that $Q_1 \cap R = P_1$, $Q_2 \cap R = P_2$. We will show that $Q_1 \subseteq Q_2$. We have four cases.

Case I: If $Q_1, Q_2 \in \mathscr{T}'$, then $P_1, P_2 \in (\mathscr{T}')^c$. Since $\mathscr{T}' \cong (\mathscr{T}')^c$, $Q_1 \subseteq Q_2$.

Case II: If $Q_1 \notin \mathscr{T}'$ and $Q_2 \notin \mathscr{T}'$, then $Q_1, Q_2 \in \mathscr{B}'_\beta(Q_0)$. So $P_1, P_2 \in \mathscr{B}_\beta(P_0)$. Since $\mathscr{B}'_\beta(Q_0) \cong \mathscr{B}_\beta(P_0)$, $Q_1 \subseteq Q_2$.

Case III: If $Q_1 \in \mathscr{T}'$ and $Q_2 \notin \mathscr{T}'$, then $Q_2 \in \mathscr{B}'_\beta(Q_0)$ and hence $P_0 = Q_0 \cap R \subseteq Q_2 \cap R = P_2$, $P_1 = Q_1 \cap R \in (\mathscr{T}')^c$. Since $P_1 \subseteq P_2$ and $P_2 \in \mathscr{B}_\beta$, $P_1 \in \mathscr{B}_\beta$ by Lemma 2. Then $P_1 \in (\mathscr{T}')^c \cap \mathscr{B}_\beta = (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha) \cap \mathscr{B}_\beta$ and hence $P_1 \subseteq P_0$. Thus we have $P_1 \subseteq P_0 \subseteq P_2$. If $P_0 \in \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$, then in view of the construction of Q_0, $Q_0 \in \mathscr{T}'$ and hence $Q_1 \subseteq Q_0$. Since $\mathscr{T}' \cong (\mathscr{T}')^c$ and $Q_2 \in \mathscr{B}'_\beta(Q_0)$, we have $Q_1 \subseteq Q_0 \subseteq Q_2$. If $P_0 \notin \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$, then $Q_0 = \bigcup_{\gamma\in\Theta} Q_\gamma \notin \mathscr{T}'$. Since $P_1 \subseteq P_0$ and $P_1 \in \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$ while $P_0 \notin \bigcup_{\alpha\in\Gamma} B_\alpha$, we have $P_1 \subset P_0$. From $P_0 = \bigcup_{\gamma\in\Theta} P_\gamma$, it follows that there exists $\gamma \in \Theta$ such that $P_1 \subseteq P_\gamma$ (note that P_1 and each P_γ are comparable since $P_1 \subseteq P_0 \subseteq P$ implies $P_1 \in \mathscr{B}_\beta$). Since $Q_1, Q_\gamma \in \mathscr{T}'$ and $\mathscr{T}' \cong (\mathscr{T}')^c$, we have $Q_1 \subseteq Q_\gamma$. Hence $Q_1 \subseteq Q_\gamma \subseteq \bigcup_{\gamma\in\Theta} Q_\gamma = Q_0$. Since $Q_2 \in \mathscr{B}'_\beta(Q_0)$, $Q_1 \subseteq Q_2$.

Case IV: If $Q_1 \notin \mathscr{T}'$ and $Q_2 \in \mathscr{T}'$, then $P_1 \in \mathscr{B}_\beta \setminus (\mathscr{T}')^c$.

Since $P_2 = Q_2 \cap R$ and $Q_2 \in \mathscr{T}'$, $P_2 \in (\mathscr{T}')^c = \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha$. So $P_2 \in \mathscr{B}_\alpha$ for some α. Since $P_1 \subseteq P_2$, we have $P_1 \in \mathscr{B}_\alpha$ by Lemma 2. Hence $P_1 \in \bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha = (\mathscr{T}')^c$. It is a contradiction to that $P_1 \in \mathscr{B}_\beta \setminus (\mathscr{T}')^c$. Thus, Case IV never occurs. Then from Cases I–III, we deduce that $Q_1 \subseteq Q_2$. Thus, the inverse function of the contraction function preserves the partial order, namely inclusion.

Hence $\mathscr{T}'_1 \cong (\mathscr{T}'_1)^c$. Since $(\mathscr{T}'_1)^c$ is a tree and $\mathscr{T}'_1$ is isomorphic to $(\mathscr{T}_1)^c$, $\mathscr{T}'_1$ is also a tree.

Therefore $(\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha \cup \mathscr{B}_\beta, \mathscr{T}'_1) > (\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha, \mathscr{T}')$, which contradicts that $(\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha, \mathscr{T}')$ is a maximal element in $\mathscr{S}$. Hence $\bigcup_{\alpha\in\Gamma} \mathscr{B}_\alpha = \mathscr{T}$. $\square$

3. An SCLO extension and a GU extension are the same

Now we prove that the SCLO extension is identical with the GU extension.

Lemma 8. *Let $D \subseteq B$ be a GU extension of integral domains.*

Let $\mathscr{C} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a chain in $Spec(D)$ and $S = \bigcup_{\alpha\in\Lambda} S_\alpha$, where $S_\alpha = (D \setminus P_\alpha)^{-1}P_\alpha$. Then $\langle S\rangle := SB[S]$ is a proper ideal of $B[S]$.

Proof. Suppose that $\langle S\rangle := SB[S] = (1)$. Then $1 = s_1'b_1' + s_2'b_2' + \cdots + s_m'b_m'$ for some $s_i' \in S$ and $b_i' \in B[S]$. Since S is closed under multiplication [12, Lemma 1 (1)], $1 = s_1b_1 + \cdots + s_nb_n$ for some $s_i \in S$ and $b_i \in B$. Let $s_i \in (D\setminus P_i)^{-1}P_i$. Since $\{P_\alpha\}_{\alpha\in\Lambda}$ is a chain, we may assume that $P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_1$. By the GU-property, there exists $Q_n' \subseteq Q_{n-1}' \subseteq \cdots \subseteq Q_1'$ in $Spec(B)$ lying over $P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_1$. By [10, Corollary 19.7], there exists a valuation overring V of B and a chain of prime ideals $Q_n \subseteq Q_{n-1} \subseteq \cdots \subseteq Q_1$ of V lying over $Q_n' \subseteq Q_{n-1}' \subseteq \cdots \subseteq Q_1'$.

Put $s_i := g_i/f_i$, where $g_i \in P_i$ and $f_i \in D\setminus P_i$. Since $g_i \in Q_i$, $f_i/g_i \notin V$ (for otherwise $f_i \in g_iV \cap D \subseteq Q_i \cap D = P_i$, a contradiction). Since V is a valuation overring of B, $s_i = g_i/f_i \in V$. Then $s_if_i = g_i \in Q_i$. Hence $s_i \in Q_i$ since $f_i \notin Q_i$. So $b_is_i \in Q_i \subseteq Q_1$ for each i. From this it follows that $1 = s_1b_1 + \cdots + s_nb_n \in Q_1$, a contradiction. □

Lemma 9. *Let $D \subseteq B$ be a GU extension of integral domains and F be a field containing B. Let $\mathscr{C} = \{P_\alpha\}_{\alpha\in\Lambda}$ be a chain in $Spec(D)$. Then there exists a valuation domain V on F containing B and a chain $\mathscr{C}'$ in $Spec(V)$ lying over $\mathscr{C}$.*

Proof. By Lemma 8, $\langle S\rangle := SB[S]$ is a proper ideal of $B[S]$. By [10, Proposition (19.6)], there exists a valuation domain (V,M) on F containing $B[S]$ such that $\langle S\rangle \subseteq M\cap B[S]$. Then we proceed as in the proof of [12, Theorem] to show that $\sqrt{P_\alpha V}\cap D = P_\alpha$. Suppose that it fails. We choose an element $f \in \sqrt{P_\alpha V} \cap D\setminus P_\alpha$. Then $f \in D\setminus P_\alpha$ and $f^n \in P_\alpha V, n \in \mathbb{N}$. So $f^n/g \in V$, $g \in P_\alpha$. This implies $g/f^n \in S$ is a unit in V. This contradicts the fact that $\langle S\rangle \subseteq M$. Hence $\sqrt{P_\alpha V} \cap D = P_\alpha$. Furthermore $\sqrt{P_\alpha V}$ is a prime ideal in V [10, Theorem 17.1]. □

Theorem 10. *Let $D \subseteq B$ be a GU extension of integral domains. Then for each chain of prime ideals of D, there exists a chain of prime ideals of B lying over the given chain.*

Proof. Let $\mathscr{C}$ be an arbitrary chain of prime ideals of D and K be the quotient field of B. By Lemma 9, there exists a valuation domain V on K containing B and a chain $\mathscr{C}''$ in $Spec(V)$ lying over $\mathscr{C}$. Then the contraction $(\mathscr{C}'')^c$ of $\mathscr{C}''$ on $Spec(B)$ is the desired chain of prime ideals of B, which is lying over $\mathscr{C}$. □

Corollary 11. *Let $D \subseteq B$ be a GU extension of commutative rings. Then $D \subseteq B$ is an SCLO extension: if $\mathscr{C}$ is a chain of prime ideals of D with the initial element P and Q is a prime ideal of B lying over P, then there exists a chain $\mathscr{C}'$ of prime ideals of B lying over $\mathscr{C}$ whose initial element is Q.*

Proof. Let $\bar{\mathscr{C}}$ be the image of the chain $\mathscr{C}$ in D/P. Since $D/P \subseteq B/Q$ is a GU extension of integral domains, there exists a chain $\mathscr{C}''$ of prime ideals of B/Q lying over $\bar{\mathscr{C}}$ (Theorem 10). Let ψ be the canonical homomorphism: $B \to B/Q$. Then $\psi^{-1}(\mathscr{C}'')$ is a chain of prime ideals of B lying over $\mathscr{C}$. Finally $\mathscr{C}' := \psi^{-1}(\mathscr{C}'') \cup \{Q\}$ is the desired chain in $Spec(B)$ lying over $\mathscr{C}$ whose initial element is Q (note that every element in $\psi^{-1}(\mathscr{C}'')$ is bigger than or equal to Q). $\square$

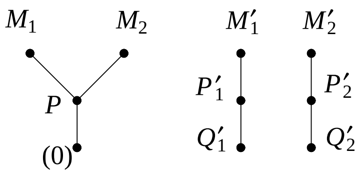
Corollary 12. *An extension $R \subseteq D$ of commutative rings has the going-up property if and only if the extension $R \subseteq D$ has the SCLO-property.*

Remark. Obviously, an extension $R \subseteq D$, which has the SCLO-property, has the CLO-property. But an extension $R \subseteq D$ with the CLO-property need not have the SCLO-property (going-up property). For example, the polynomial extension $R \subseteq R[x]$, which is a CLO extension, does not satisfy the going-up property if R has Krull-dimension $\geqslant 1$. But in the extension $R \subseteq R[x]$, any tree $\mathscr{T}$ in $Spec(R)$ can be embedded in a tree $\mathscr{T}'$ in $Spec(R[x])$.

Now we give an extension $R \subseteq D$ of commutative rings, which is a CLO extension, such that the extension $R \subseteq D$ does not satisfy the going-up property and $Spec(R)$ has a tree $\mathscr{T}$ that cannot be embedded in a tree $\mathscr{T}'$ in $Spec(D)$.

Example 13. (*A CLO extension that is not a GU extension and a tree of prime ideals that cannot be embedded in the extension*)
(1) It is known that any finite partially ordered set can be realized as the spectrum of a commutative ring with identity [14]. If the poset is a finite tree with the smallest element, then the commutative ring can further be arranged to be a Prüfer domain [14, Theorem 3.1]. Let R be an integral domain such that $Spec(R) = \{(0), P, M_1, M_2\}$, where M_1 and M_2 are the maximal ideals of R and $(0) \subset P \subset M_1 \cap M_2$. Consider $D := R_{M_1} \oplus R_{M_2}$. Then D is a commutative ring with identity such that $M_1R_{M_1} \oplus R_{M_2}$ and $R_{M_1} \oplus M_2R_{M_2}$ are the maximal ideals of D. We can identify R as the subring $\{(r,r) | r \in R\}$ of D.

Consider the diagrams of $Spec(R)$ and $Spec(D)$:



Spec (R) *Spec* (D)

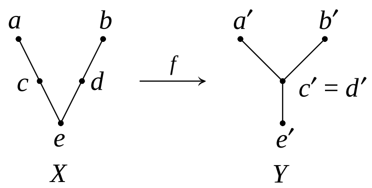
where $M_1' = M_1R_{M_1} \oplus R_{M_2}$, $M_2' = R_{M_1} \oplus M_2R_{M_2}$, $P_1' = PR_{M_1} \oplus R_{M_2}$, $P_2' = R_{M_1} \oplus PR_{M_2}$, $Q_1' = (0) \oplus R_{M_2}$, and $Q_2' = R_{M_1} \oplus (0)$.

Note that $M_1' \cap R = M_1$, $M_2' \cap R = M_2$, $P_1' \cap R = P = P_2' \cap R$, and $Q_1' \cap R = (0) = Q_2' \cap R$. Hence $R \subset D$ satisfies the CLO-property. But the going-up property does not hold. The tree $Spec(R)$ cannot be embedded in $Spec(D)$.

Another example can be constructed as following.

(2) In [11], Hochster studied the (contravariant) functor *Spec* from the category of commutative rings with identity to the category of topological spaces and continuous maps. Hochster called a topological space spectral if it is T_0 and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every irreducible closed subset has a generic point. Note that $Spec(R)$ is spectral under the Zariski-topology for any commutative ring R with identity [2, pp. 14–15, Exercise 15–20]. Hochster also called a continuous map of spectral spaces spectral if the inverse images of quasi-compact open subsets are quasi-compact. In [11, Theorem 6], he showed that every spectral space and spectral map is isomorphic to one in the image of *Spec*. As in [14], we can topologize a finite poset X by letting a subbasis for the closed sets be $Cl(x)=\{y\in X\,|\,y\geqslant x\}$. Note that a finite poset X is spectral.

Consider two finite posets X, Y and an order-preserving map $f:X\to Y$ as follows:



where $f(a)=a'$, $f(b)=b'$, $f(c)=f(d)=c'=d'$, and $f(e)=e'$. Here points represent elements, the line segments describe the ordering, and a,b,a',b' are maximal elements. Note that f is spectral. By [11, Theorem 6], we can get two commutative rings R, D with identity and a ring homomorphism $F:R\to D$ such that $Spec(R)=Y$, $Spec(D)=X$, and $Spec(F)=f$. Since $Spec(R)=Spec(R/(kerF))$, we may identify R as a subring of D. Then the extension $R\subseteq D$ of commutative rings is a CLO extension such that the extension $R\subseteq D$ does not satisfy the going-up property and the tree $Spec(R)$ can not be embedded in a tree in $Spec(D)$.

Remark. When $R\subseteq D$ is a GU extension, a subset of $Spec(R)$, which is not a tree, may not be embedded in $Spec(D)$. Example 18 will exhibit such an example.

Corollary 14 (cf. Dobbs and Fontana [7, Theorem 3]). *Let $R\subseteq D$ be an extension of commutative rings with the going-up property. Let $\mathscr{T}\subseteq Spec(R)$ be a tree. Then $\mathscr{T}$ can be embedded in a tree $\mathscr{T}'\subseteq Spec(D)$.*

Proof. This follows from Theorem 7 and Corollary 12. □

Remark. Example 13 shows that Corollary 14 does not hold if $R\subseteq D$ is not a going-up extension.

Corollary 15. *Let R' be the integral closure of R and $\mathscr{T}$ be a tree in $Spec(R)$. Then $\mathscr{T}$ can be embedded into $Spec(R')$, i.e., there exists an isomorphic tree in $Spec(R')$ lying over $\mathscr{T}$.*

Proof. The extension $R\subseteq R'$ satisfies the going-up property. □

It is well known that the prime spectrum of a Prüfer domain is a tree and the integral closure of a Prüfer domain is a Prüfer domain. In fact, the prime spectrum of the smaller Prüfer domain can be realized as a subtree of the bigger Prüfer domain, which is demonstrated in the next result.

Corollary 16. *Let R be a Prüfer domain and R' be the integral closure of R in some extension field of the quotient field of R. Then $Spec(R)$ can be embedded in $Spec(R')$.*

4. Every tree of prime ideals is isomorphic to a tree of prime ideals in a Prüfer domain

Theorem 17. *Let R be an integral domain. Then there exists a Prüfer domain* (*in fact, a Bézout domain*) *D such that every tree in $Spec(R)$ can be embedded in $Spec(D)$.*

Proof. Let $D := (R')^b$ be the Kronecker function ring of R' (with respect to completion), where R' is the integral closure of R. It is well known that D is a Bézout domain. Also it is known that D is a going-up extension of R [6, Theorem 11(a)]. The conclusion follows from Theorem 7 and Corollary 12. □

5. Contraction of a tree of prime ideals

Now we consider a question of Dobbs and Fontana: given an extension $A \subsetneq B$ of commutative rings and a tree $\mathscr{T} \subsetneq Spec(B)$, what are necessary and sufficient conditions that $(\mathscr{T}')^c = \{Q \cap A \mid Q \in \mathscr{T}\}$ be a tree in $Spec(A)$? We give an example which helps us to find a sufficient condition. Before we give an example, for a commutative ring R we define a diamond subset of $Spec(R)$. A subset $\mathscr{T} \subseteq Spec(R)$ is a diamond set if there is an element of $\mathscr{T}$ containing incomparable elements of $\mathscr{T}$.

In the following example, we show that in an integral extension, a subset of prime ideals, which is not a tree, may not be embedded in the spectrum of the extension.

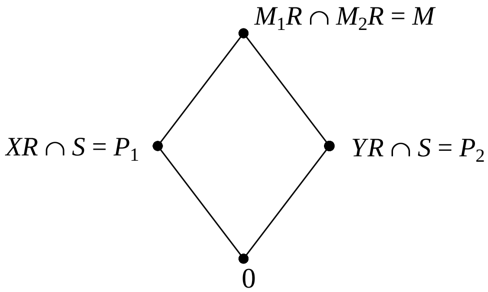
Example 18. (*An example of an integral extension $S \subseteq R$ with a subset $\mathscr{D} \subseteq Spec(S)$ such that $\mathscr{D}$ cannot be embedded in $Spec(R)$*)
Consider $K[X,Y]$, where K is a field and X, Y are indeterminates. $M_1 = (X, Y+1), M_2 = (X+1, Y)$ are maximal ideals. Let $R = K[X,Y]_N$, where $N = K[X,Y] \setminus (M_1 \cup M_2)$. Then $\{M_1R, M_2R\}$ is the set of maximal ideals of R [10, Propositon 4.8]. So R is a two-dimensional integrally closed Noetherian domain with $\mathrm{Max}(R) = \{M_1R, M_2R\}$. Let $S = K + (M_1R \cap M_2R)$. By the Chinese remainder theorem, $R/(M_1R \cap M_2R) \simeq R/M_1R \oplus R/M_2R \simeq K \oplus K$.

Consider the diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S/(M_1R \cap M_2R) \simeq K \arrow[d, "i"] \\ R \arrow[r, "\varphi"] & R/(M_1R \cap M_2R) \simeq K \oplus K \end{tikzcd}$$

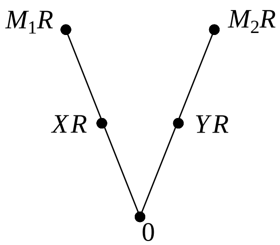
It is a pullback diagram where i is the inclusion map and φ is the canonical surjection. Note that S is also a Notherian domain (Eakin's Theorem or [9, Proposition 1.1.7]). Since R and S have the common ideal $M_1R \cap M_2R$, R and S have common quotient field. Let S' be the integral closure of S and S^* be the complete integral closure of S. Then $S' = S^* = R$ [9, Lemma 1.1.4 (10)].

Consider the diamond diagram $\mathscr{D}$ in $Spec(S)$:



Note that $M = M_1R \cap S = M_2R \cap S$ and $P_1, P_2 \subsetneq M$. (Since R is integral over S, $M_1R \cap S$ and $M_2R \cap S$ are maximal ideals of S containing $M_1R \cap M_2R = M$. Since M is a maximal ideal of S, $M = M_1R \cap S = M_2R \cap S$. Since $XR, YR \subsetneq M_1R$ and R is integral over S, $P_1 = XR \cap S \subsetneq M_1R \cap S = M$ and $P_2 = YR \cap S \subsetneq M_2R \cap S = M$.) Furthermore XR (respectively, YR) is the unique prime ideal of R such that $XR \cap S = P_1$ (respectively, $YR \cap S = P_2$) [9, Lemma 1.1.4 (3)].

Now consider the following tree $\mathscr{T}$ in $Spec(R)$:



Note that $YR \nsubseteq M_1R$, $XR \nsubseteq M_2R$. Since R is integral over S and M is a maximal ideal of S, $\mathrm{Max}(R) = \{M_1R, M_2R\}$ is the set of prime ideals of R which are lying over M. Hence the diamond diagram in $Spec(S)$ cannot be covered by a diamond in $Spec(R)$, while it is covered by a tree. Note that the above tree in $Spec(R)$ contracts to a diamond subset of $Spec(S)$. Moreover the tree $\mathscr{T}$ is the unique subset of $Spec(R)$ lying over $\mathscr{D}$. So the diamond set $\mathscr{D}$ cannot be embedded in $Spec(R)$.

As seen in Example 18, in an integral extension, two incomparable prime ideals can contract to the same prime ideal. In order to get a sufficient condition for the question of Dobbs and Fontana, we consider unibranched extensions. A ring extension $A \subseteq B$ is unibranched (respectively, mated) if, for any prime P of A (respectively, for any prime P of A such that $PB \neq B$), there is exactly one prime ideal of B lying over P [3]. Note that unibranchness implies matedness under the lying-over property. The reader may consult [3,15,16] for the unibranched (mated) extensions. Note that the extension $S \subseteq R$ in Example 18 is not mated.

Proposition 19. *Let $R \subseteq D$ be mated with the going-up property and $\mathscr{T}$ be a tree in $Spec(D)$. Then*

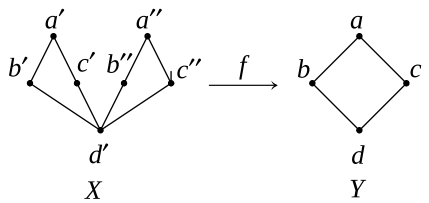
(1) *$(\mathscr{T})^c$ is also a tree in $Spec(R)$ and $\mathscr{T} \cong (\mathscr{T})^c$.*
(2) *If $Spec(R)$ is a tree, then $Spec(D)$ is also a tree and moreover $Spec(D) \cong Spec(R)$.*

Proof. (1) Let $P_1, P_2, P_0 \in (\mathscr{T})^c$ and $P_1, P_2 \subseteq P_0$. We can choose Q_1, Q_2, and Q_0 in $\mathscr{T}$ such that $Q_i \cap R = P_i$ for $i = 0, 1, 2$. Since $P_1 \subseteq P_0$, $Q_1 \cap R = P_1$ and $R \subseteq D$ has SCLO-property, there exists a prime ideal Q in D such that $Q \cap R = P_0$ and $Q_1 \subseteq Q$. Since the extension is mated, $Q = Q_0$. By the same argument, we have $Q_1, Q_2 \subseteq Q_0$. Since $\mathscr{T}$ is tree, Q_1, Q_2 are comparable. Hence P_1, P_2 are comparable. Thus $(\mathscr{T})^c$ is also a tree in $Spec(R)$. Since the extension $R \subseteq D$ is mated, $\mathscr{T} \cong (\mathscr{T})^c$.

(2) By Corollary 14, $Spec(R)$ can be embedded into $Spec(D)$. Since D is a mated extension of R, $Spec(R) \cong Spec(D)$. $\square$

Remark. It is easily shown that a mated extension satisfies the going-down property under the going-up property. But as in shown in [16, Example], the converse does not hold. Although the converse fails to be true, it is natural to ask whether the Proposition 19 is true for an extension $R \subseteq D$ with the going-up and going-down properties. The answer is negative, which is illustrated in the following example.

Example 20. We give an extension $R \subseteq D$ of commutative rings, which has the going-up and going-down properties, such that $Spec(D)$ has a tree $\mathscr{T}'$ and $(\mathscr{T}')^c$ is not a tree in $Spec(R)$. We adopt the method used in Example 13 (2). Consider two finite posets X, Y and an order-preserving map $f : X \to Y$ as follows:



where $f(a') = f(a'') = a$, $f(b') = f(b'') = b$, $f(c') = f(c'') = c$, and $f(d') = d$. Here points represent elements, the line segments describe the ordering, and a, a', a'' are maximal elements. Note that f is spectral. By [11, Theorem 6], we can get two commutative rings R, D with identity and a ring homomorphism $F : R \to D$ such that $Spec(R) = Y$, $Spec(D) = X$, and $Spec(F) = f$. Since $Spec(R) = Spec(R/(kerF))$, we identify R as a subring of D. Then the extension $R \subseteq D$ has the going-up and going-down properties. Let $\mathscr{T}' := \{a', a'', b', c'', d'\}$. Then $\mathscr{T}'$ is a tree in $Spec(D)$, but $(\mathscr{T}')^c = Y$ is not a tree.

Finally, although there is an integral domain [7, Example 8(b)] which gives a negative answer to the question of Anderson [1, Problem 7(b)], we briefly mention the property of an integral domain which satisfies the question of Anderson positively. We say that a domain R has PT-property if, for a given tree $\mathscr{T} \subseteq Spec(R)$, there exists a Prüfer overring $P(R)$ with the corresponding tree $\mathscr{T}'$ such that $(\mathscr{T}')^c = \mathscr{T}$ and $\mathscr{T}' \cong \mathscr{T}$. We end this paper by proving that a mated integral extension of a domain with PT-property also has PT-property.

Proposition 21. *If R has PT-property and $R \subseteq D$ is an integral and mated extension of integral domains, then D has PT-property.*

Proof. Let $\mathscr{S}$ be a given tree in $Spec(D)$. Since the extension is integral and mated, $\mathscr{T} = \mathscr{S} \cap R$ is a tree in $Spec(R)$ and $\mathscr{T} \cong \mathscr{S}$ by Proposition 18. Since R has PT-property, there exists a Prüfer overring $P(R)$ of R with a tree $\mathscr{T}'$ such that $\mathscr{T}' \cap R = \mathscr{T}$ and $\mathscr{T}' \cong \mathscr{T}$. Let $P(D)$ be an integral closure of $P(R)$ in L, the quotient field of D. Then $P(D)$ is a Prüfer overring of D. Since $P(R) \subseteq P(D)$ is integral, Theorem 7 and Proposition 5 imply that there exists a tree $\mathscr{S}'$ in $Spec(P(D))$ such that $\mathscr{S}' \cap P(R) = \mathscr{T}'$ and $\mathscr{S}' \cong \mathscr{T}'$. Then $\mathscr{S}' \cap D = \mathscr{S}$ and $\mathscr{S}' \cong \mathscr{S}$. Thus D has PT-property. □

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